Problem 2.18

Taylor's theorem states that, for any reasonable function $f(x)$, the value of f at a point $(x+\delta)$ can be expressed as an infinite series involving f and its derivatives at the point x:

$$f(x+\delta) = f(x) + f'(x)\delta + \frac{1}{2!}f''(x)\delta^2 + \frac{1}{3!}f'''(x)\delta^3 + \cdots \tag{2.87}$$

where the primes denote successive derivatives of $f(x)$. (Depending on the function this series may converge for *any* increment δ or only for values of δ less than some nonzero "radius of convergence.") This theorem is enormously useful, especially for small values of δ, when the first one or two terms of the series are often an excellent approximation.[11] **(a)** Find the Taylor series for $\ln(1+\delta)$. **(b)** Do the same for $\cos\delta$. **(c)** Likewise $\sin\delta$. **(d)** And e^δ.

Solution

Part (a)

Use the given formula to expand the function.

$$\begin{aligned}
\ln(1+\delta) &= \ln 1 + \frac{d}{dx}(\ln x)\bigg|_{x=1}\delta + \frac{1}{2!}\frac{d^2}{dx^2}(\ln x)\bigg|_{x=1}\delta^2 + \frac{1}{3!}\frac{d^3}{dx^3}(\ln x)\bigg|_{x=1}\delta^3 + \frac{1}{4!}\frac{d^4}{dx^4}(\ln x)\bigg|_{x=1}\delta^4 + \cdots \\
&= 0 + \left(\frac{1}{x}\right)\bigg|_{x=1}\delta + \frac{1}{2!}\left(-\frac{1}{x^2}\right)\bigg|_{x=1}\delta^2 + \frac{1}{3!}\left(\frac{2}{x^3}\right)\bigg|_{x=1}\delta^3 + \frac{1}{4!}\left(-\frac{6}{x^4}\right)\bigg|_{x=1}\delta^4 + \cdots \\
&= 0 + (1)\delta + \frac{1}{2}(-1)\delta^2 + \frac{1}{6}(2)\delta^3 + \frac{1}{24}(-6)\delta^4 + \cdots \\
&= \delta - \frac{1}{2}\delta^2 + \frac{1}{3}\delta^3 - \frac{1}{4}\delta^4 + \cdots
\end{aligned}$$

Part (b)

Use the given formula to expand the function.

$$\begin{aligned}
\cos\delta &= \cos(0+\delta) \\
&= \cos 0 + \frac{d}{dx}(\cos x)\bigg|_{x=0}\delta + \frac{1}{2!}\frac{d^2}{dx^2}(\cos x)\bigg|_{x=0}\delta^2 + \frac{1}{3!}\frac{d^3}{dx^3}(\cos x)\bigg|_{x=0}\delta^3 + \frac{1}{4!}\frac{d^4}{dx^4}(\cos x)\bigg|_{x=0}\delta^4 + \cdots \\
&= \cos 0 + (-\sin x)\bigg|_{x=0}\delta + \frac{1}{2!}(-\cos x)\bigg|_{x=0}\delta^2 + \frac{1}{3!}(\sin x)\bigg|_{x=0}\delta^3 + \frac{1}{4!}(\cos x)\bigg|_{x=0}\delta^4 + \cdots \\
&= \cos 0 + (0)\delta + \frac{1}{2}(-1)\delta^2 + \frac{1}{6}(0)\delta^3 + \frac{1}{24}(1)\delta^4 + \cdots = 1 - \frac{1}{2}\delta^2 + \frac{1}{24}\delta^4 + \cdots
\end{aligned}$$

[11] For more details on Taylor's series see, for example, Mary Boas, *Mathematical Methods in the Physical Sciences* (Wiley, 1983), p. 22 or Donald McQuarrie, *Mathematical Methods for Scientists and Engineers* (University Science Books, 2003), p. 94.

Part (c)

Use the given formula to expand the function.

$$
\begin{aligned}
\sin\delta &= \sin(0+\delta) \\
&= \sin 0 + \frac{d}{dx}(\sin x)\bigg|_{x=0}\delta + \frac{1}{2!}\,\frac{d^2}{dx^2}(\sin x)\bigg|_{x=0}\delta^2 + \frac{1}{3!}\,\frac{d^3}{dx^3}(\sin x)\bigg|_{x=0}\delta^3 + \frac{1}{4!}\,\frac{d^4}{dx^4}(\sin x)\bigg|_{x=0}\delta^4 + \cdots \\
&= \sin 0 + (\cos x)\bigg|_{x=0}\delta + \frac{1}{2!}(-\sin x)\bigg|_{x=0}\delta^2 + \frac{1}{3!}(-\cos x)\bigg|_{x=0}\delta^3 + \frac{1}{4!}(\sin x)\bigg|_{x=0}\delta^4 + \frac{1}{5!}(\cos x)\bigg|_{x=0}\delta^5 + \cdots \\
&= \sin 0 + (1)\delta + \frac{1}{2}(0)\delta^2 + \frac{1}{6}(-1)\delta^3 + \frac{1}{24}(0)\delta^4 + \frac{1}{120}(1)\delta^5 + \cdots \\
&= \delta - \frac{1}{6}\delta^3 + \frac{1}{120}\delta^5 + \cdots
\end{aligned}
$$

Part (d)

Use the given formula to expand the function.

$$
\begin{aligned}
e^\delta &= e^{0+\delta} \\
&= e^0 + \frac{d}{dx}(e^x)\bigg|_{x=0}\delta + \frac{1}{2!}\,\frac{d^2}{dx^2}(e^x)\bigg|_{x=0}\delta^2 + \frac{1}{3!}\,\frac{d^3}{dx^3}(e^x)\bigg|_{x=0}\delta^3 + \frac{1}{4!}\,\frac{d^4}{dx^4}(e^x)\bigg|_{x=0}\delta^4 + \cdots \\
&= e^0 + (e^x)\bigg|_{x=0}\delta + \frac{1}{2!}(e^x)\bigg|_{x=0}\delta^2 + \frac{1}{3!}(e^x)\bigg|_{x=0}\delta^3 + \frac{1}{4!}(e^x)\bigg|_{x=0}\delta^4 + \frac{1}{5!}(e^x)\bigg|_{x=0}\delta^5 + \cdots \\
&= e^0 + (1)\delta + \frac{1}{2}(1)\delta^2 + \frac{1}{6}(1)\delta^3 + \frac{1}{24}(1)\delta^4 + \frac{1}{120}(1)\delta^5 + \cdots \\
&= 1 + \delta + \frac{1}{2}\delta^2 + \frac{1}{6}\delta^3 + \frac{1}{24}\delta^4 + \frac{1}{120}\delta^5 + \cdots
\end{aligned}
$$